

# Lecture 17

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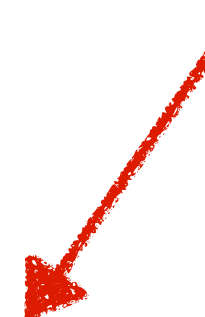
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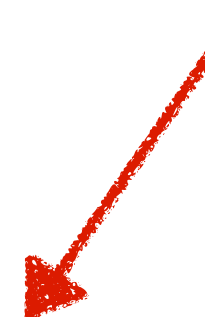
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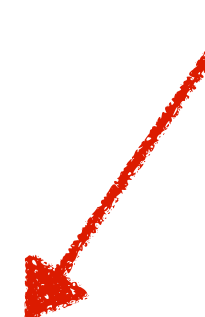
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**Certificate:**  $v_1 : path_i(v_1), v_2 : path_i(v_2), \dots, v_{|C_i|} : path_i(v_{|C_i|})$ , where  $v_j < v_{j+1}$ .

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**Corollary:** For every space-constructible function  $S(n) \geq \log n$ ,

$$NSPACE(S(n)) = coNSPACE(S(n)).$$